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Superadditivity and derivative of operator functions

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Abstract

We will show that $\sum_{i\neq j}A_iA_j \geq 0$ for bounded operators $A_i \geq 0$ $(i=1,2,\cdots,n)$ if and only if $g(\sum_i A_i) \geq \sum_i g(A_i)$ for every operator convex function $g(t)$ on $[0,\infty)$ with $g(0)\leq 0$. Let $A, B\geq 0$ and A be invertible. Then we will observe that the Fréchet derivative $Dg(sA)(B)$ is increasing on $0<s<\infty$ for every operator convex function $g(t)$ on $(0,\infty)$ if and only if $AB+BA\geq 0$.

1 Introduction

Let I be an interval of the real axis and $f(t)$ a real continuous function defined on I. For a bounded Hermitian operator (or matrix) A on a Hilbert space whose spectrum is in I, $f(A)$ stands for the ordinary functional calculus. f is called an *operator monotone* (*or operator decreasing*) function on I if $f(A)\leq f(B)$ (or $f(A)\geq f(B)$) whenever $A\leq B$. It is evident that if $f(t)$ is operator monotone in the interior of I and continuous on I, then $f(t)$ is operator monotone on I itself. It is an essential fact that $f_\lambda(t):=\frac{\lambda t}{\lambda+t}$ is operator monotone on $(-\infty,-\lambda)$ and on $(-\lambda,\infty)$ for each λ. It is also well-known that t^a $(0<a\leq 1)$ is operator monotone on $[0,\infty)$ and so is $\log t$ on $(0,\infty)$. A continuous function g defined on I is called an *operator convex function* on I if $g(sA+(1-s)B)\leq sg(A)+(1-s)g(B)$ for every $0<s<1$ and for every pair of bounded Hermitian operators A and B whose spectra are both in I. An *operator concave function* is likewise defined. t^a $(1\leq a\leq 2)$ and $t\log t$ are both operator convex on $[0,\infty)$. For further details we refer the reader to[2,8]. It has been well-known that a non-negative continuous function $f(t)$ on $[0,\infty)$ is operator monotone if and only if $f(t)$ is operator concave. One of the authors [12,15] (cf.[7]) extended this as follows:
A continuous function $f(t)$ defined on an infinite interval (a,∞) is operator monotone if and only if $f(t)$ is operator concave and $f(\infty)>-\infty$.

Let $h(t)$ be a non-negative concave (not necessarily operator concave) function on $[0,\infty)$. Since $h(t)$ is increasing and $h(t)/t$ is decreasing, $h(t)$ is subadditive, namely $h(a+b) \leq h(a)+h(b)$. Bourin and M. Uchiyama[3] have shown the following theorems.

Theorem 1.1. *Let $A, B \geq 0$ and let $f : [0,\infty) \to [0,\infty)$ be a concave function. Then, for every unitarily invariant norm $\|\cdot\|$,*

$$\|f(A+B)\| \leq \|f(A)+f(B)\|.$$

Theorem 1.2. *Let $A, B \geq 0$ and let $g : [0,\infty) \to [0,\infty)$ be a convex function with $g(0)=0$. Then, for every unitarily invariant norm $\|\cdot\|$,*

$$\|g(A+B)\| \geq \|g(A)+g(B)\|.$$

Moslehian and Najafi [9] have shown the following theorem.

Theorem 1.3. *For $A, B \geq 0$, $AB+BA \geq 0$ is equivalent to*

$$f(A+B) \leq f(A)+f(B)$$

for any non-negative operator monotone function $f \geq 0$ on $[0,\infty)$.

We will show that if $\sum_{i \neq j} A_i A_j \geq 0$ for bounded self-adjoint operators A_i $(i = 1, 2, \cdots, n)$, then for every operator monotone function $f(t) \geq 0$ on $[0,\infty)$

$$f(A_1 + \cdots + A_n) \leq f(A_1) + \cdots + f(A_n),$$

and for every operator convex function with $g(0) \leq 0$

$$g(A_1 + \cdots + A_n) \geq g(A_1) + \cdots + g(A_n).$$

For Theorem 1.1 and Theorem 1.2, we can extend the results to finitely many operators by using the exactly the same proof. But, as for Theorem 3, we have to use to different proof to extend the results to finitely many operators.

If $h(t)$ is a C^1-function defined on an open interval, then the matrix function $h(X)$ is Fréchet differentiable and the derivative $Dh(A)(B)$ equals the Gateaux derivative $\frac{d}{dt}h(A+tB)|_{t=0}$. It is known that if $f(t)$ is operator monotone, the Fréchet derivative $Df(A)$ is a positive linear mapping (cf.[2]). It is easy to see $g(t)$ is operator convex if and only if $\langle g(A+t(B-A))x, x\rangle$ is a convex function on $0 \leq t \leq 1$ for all A, B and for all vectors x .Therefore, if $g \in C^1$ is operator convex, then

$$g(B) \geq g(A) + Dg(A)(B-A). \tag{1}$$

Because

$$\langle g(A+t(B-A))x, x\rangle = \langle g((1-t)A+tB)x, x\rangle \leq \langle (1-t)g(A)x, x\rangle + \langle tg(B)x, x\rangle.$$

At $t = 0$, an equality holds, and the derivative of the right hand side is independent of t. Then

$$\frac{d}{dt}\langle g(A+t(B-A))x, x\rangle|_{t=0} \leq \langle -g(A)x, x\rangle + \langle g(B)x, x\rangle.$$

$$Dg(A)(B-A) \leq -g(A) + g(B).$$

Let $A, B \geq 0$ and A be invertible. Then we will prove that the Fréchet derivative $Dg(sA)(B)$ is increasing on $0 < s < \infty$ for all operator convex functions $g(t)$ on $(0,\infty)$ if and only if $AB+BA \geq 0$.

2 Subadditivity and superadditivity

We extend Theorem 3 to finitely many operators.

Theorem 2.1. *Let $A_i \geq 0$ $(i = 1, 2, \cdots, n)$. Then the following are equivalent:*

(i) $\sum_{i \neq j} A_i A_j \geq 0$,

(ii) *for every operator convex function $g(t)$ on $[0, \infty)$ with $g(0) \leq 0$*

$$g(A_1 + \cdots + A_n) \geq g(A_1) + \cdots + g(A_n),$$

(iii) *for every non-negative operator monotone function $f(t)$ on $[0, \infty)$*

$$f(A_1 + \cdots + A_n) \leq f(A_1) + \cdots + f(A_n).$$

Proof. We show that (i) implies (ii). We first note that (i) is equivalent to $(A_1 + \cdots + A_n)^2 \geq A_1^2 + \cdots + A_n^2$, which does not imply $(A_1 + \cdots + A_{n-1})^2 \geq A_1^2 + \cdots + A_{n-1}^2$. We may assume that the spectra of A_i are in $(0, \infty)$ without loss of generality. The function $g(t)/t$ is operator monotone on $(0, \infty)$ [6] and hence operator concave, although it is not necessarily non-negative. We therefore get

$$\begin{aligned} \frac{g}{t}\Big(\sum_i A_i\Big) &= \frac{g}{t}\left(\Big(\sum_i A_i\Big)^{-1/2}\Big(\sum_j A_j\Big)^2\Big(\sum_i A_i\Big)^{-1/2}\right) \\ &\geq \frac{g}{t}\left(\Big(\sum_i A_i\Big)^{-1/2}\Big(\sum_j A_j^2\Big)\Big(\sum_i A_i\Big)^{-1/2}\right). \end{aligned}$$

Since $g(t)/t$ is operator concave on $(0, \infty)$ and

$$\Big(\sum_i A_i\Big)^{-1/2} A_1 \Big(\sum_i A_i\Big)^{-1/2} + \cdots + \Big(\sum_i A_i\Big)^{-1/2} A_n \Big(\sum_i A_i\Big)^{-1/2} = 1$$

by Choi's inequality [4] (or [5], also see Theorem 3.3 of [13])

$$\begin{aligned} &\frac{g}{t}\left(\Big(\sum_i A_i\Big)^{-1/2}\Big(\sum_j A_j^2\Big)\Big(\sum_i A_i\Big)^{-1/2}\right) \\ &= \frac{g}{t}\left(\sum_j\left(\Big(\sum_i A_i\Big)^{-1/2} A_j^{1/2} A_j A_j^{1/2}\Big(\sum_i A_i\Big)^{-1/2}\right)\right) \\ &\geq \sum_j\left(\Big(\sum_i A_i\Big)^{-1/2} A_j^{1/2}\frac{g}{t}(A_j) A_j^{1/2}\Big(\sum_i A_i\Big)^{-1/2}\right) \\ &= \Big(\sum_i A_i\Big)^{-1/2}\Big(\sum_j g(A_j)\Big)\Big(\sum_i A_i\Big)^{-1/2}. \end{aligned}$$

By combining the above inequalities,we get

$$\frac{g}{t}\Big(\sum_i A_i\Big) \geq \Big(\sum_i A_i\Big)^{-1/2}\Big(\sum_j g(A_j)\Big)\Big(\sum_i A_i\Big)^{-1/2}.$$

(ii) arises by multiplying $(\sum_i A_i)^{1/2}$ from the both sides.It is evident that (ii) implies (iii), because $-f(t)$ is operator convex with $-f(0) \leq 0$. At last, we show (i) follows (iii). Since $f_\lambda(t)$ is operator monotone on $[0,\infty)$ for every $\lambda > 0$, in virtue of the decomposition $f_\lambda(t) = t - \frac{t^2}{\lambda+t}$, we obtain

$$\frac{(A_1 + \cdots + A_n)^2}{\lambda + A_1 + \cdots + A_n} \geq \frac{A_1^2}{\lambda + A_1} + \cdots + \frac{A_n^2}{\lambda + A_n}.$$

Multiply the both sides by λ and then let λ tend to ∞. This deduces (i). □

We note that (ii) directly implies (i) since t^2 itself is operator convex.

By considering $g(t) - g(0)$ and $f(t) - f(0)$, we can replace the conditions (ii) and (iii) of Theorem 2.1 with (ii)′ and (iii)′ given below, respectively.

Corollary 2.2. *Suppose $A_i \geq 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i\neq j} A_iA_j \geq 0$. Then*

(ii)′ *for every operator convex function $g(t)$ on $[0,\infty)$*

$$g(A_1 + \cdots + A_n) + (n-1)g(0)I \geq g(A_1) + \cdots + g(A_n).$$

(iii)′ *for every operator monotone function $f(t)$ on $[0,\infty)$*

$$f(A_1 + \cdots + A_n) + (n-1)f(0)I \leq f(A_1) + \cdots + f(A_n).$$

Replacing A_1, A_2, A_3 with $0I, B, C$ in (ii)′ in Corollary 2.2, we obtain

$$g(0I + B + C) + 2g(0)I \geq g(0)I + g(B) + g(C).$$

Then

$$g(0I + B + C) - g(0I + B) \geq g(0I + C) - g(0)I.$$

This is an elementary form in Lemma 3.2 stated later.

Substituting the operator convex function $t\log t$ on $[0,\infty)$ to the Theorem 2.1, we obtain the following example.

Example 2.1. If $\sum_{i\neq j} A_iA_j \geq 0$ for invertible operators $A_i \geq 0$ $(i = 1, 2, \cdots, n)$, then

$$\Big(\sum_i A_i\Big)\log\Big(\sum_i A_i\Big) \geq \sum_i A_i \log A_i.$$

Remark 2.1. We remark that if (i) does not hold, then the set of λ such that (iii) holds for $f_\lambda(t)$ is bounded, but it is not necessarily empty. For instance, let

$$A = 3\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = 4\begin{pmatrix} 8 & 10 \\ 10 & 13 \end{pmatrix}.$$

Then

$$AB + BA = 24 \begin{pmatrix} 16 & 15 \\ 15 & 13 \end{pmatrix} \not\geq 0.$$

However we have

$$\begin{aligned} -f_1(A) - f_1(B) + 1 &= (A+1)^{-1} + (B+1)^{-1} - 1 \\ = \begin{pmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} + \begin{pmatrix} 33 & 40 \\ 40 & 53 \end{pmatrix}^{-1} - 1 &= \begin{pmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} + \frac{1}{149} \begin{pmatrix} 53 & -40 \\ -40 & 33 \end{pmatrix} - 1 \\ = -\frac{1}{149} \begin{pmatrix} \frac{523}{7} & 40 \\ 40 & \frac{315}{4} \end{pmatrix} &\leq 0 \leq (A+B+1)^{-1} = -f_1(A+B) + 1. \end{aligned}$$

Hence, $f_1(A+B) \leq f_1(A) + f_1(B)$.

In [11] it was shown that $AB + BA \geq 0$ implies $A^aB + BA^a \geq 0$ for $0 < a \leq 1$ and hence $A^aB^b + B^bA^a \geq 0$ for $0 < a \leq 1$ and $0 < b \leq 1$. We therefore obtain

Corollary 2.3. *Suppose $A, B \geq 0$ and $AB + BA \geq 0$. Then for $0 < a \leq 1, 0 < b \leq 1$*

(i) $(A^a + B^b)^2 \geq A^{2a} + B^{2b}$.

(ii) *for every operator convex function $g(t)$ on $[0, \infty)$*

$$g(A^a + B^b) + g(0)I \geq g(A^a) + g(B^b).$$

(iii) *for every operator monotone function $f(t)$ on $[0, \infty)$*

$$f(A^a + B^b) + f(0)I \leq f(A^a) + f(B^b).$$

3 Fréchet derivative

Since an operator convex function $g(t)$ is convex in the usual sense, $g(a+b+c) - g(a+b) \geq g(a+c) - g(a)$ for $b, c > 0$ and $g'(t)$ is increasing. In this section we give analogous result for an operator function $g(A)$. We first give a general result.

Lemma 3.1. *Suppose $g(t)$ is operator convex on an interval. Then for A with spectrum in the interval*

$$Dg(A + sB)(B)$$

is increasing with respect to real number s as far as $g(A+sB)$ is well-defined.

Proof. From the inequality (1) it follows that for $s' < s$

$$\begin{aligned} g(A+sB) &\geq g(A+s'B) + Dg(A+s'B)(sB - s'B) \\ g(A+s'B) &\geq g(A+sB) + Dg(A+sB)(s'B - sB). \end{aligned}$$

These yield

$$Dg(A+sB)(B) \geq \frac{1}{s-s'}\left(g(A+sB)-g\left(A+s'B\right)\right) \geq Dg\left(A+s'B\right)(B).$$

This is the required result. □

From now on we deal with an operator convex function $g(t)$ or an operator monotone function $f(t)$ defined on $(0,\infty)$. We note that $g(A)$ is defined not only for invertible $A \geq 0$ but also for $A \geq 0$ if $g(0+)$ exists.

Lemma 3.2. *Let $A, B, C \geq 0$. If*

$$B(\lambda+A)^{-1}C + C(\lambda+A)^{-1}B \geq 0 \quad (\lambda > 0), \tag{2}$$

then

(i) *$f(A+B+C)-f(A+B) \leq f(A+C)-f(A)$ for every operator monotone function $f(t)$ on $(0,\infty)$ as far as $f(A)$ is bounded,*

(ii) *$g(A+B+C)-g(A+B) \geq g(A+C)-g(A)$ for every operator convex function $g(t)$ on $(0,\infty)$ as far as $g(A)$ is bounded.*

Proof. From (2), we get

$$\begin{aligned}(B+C)(\lambda+A)^{-1}(B+C) &\geq B(\lambda+A)^{-1}B + C(\lambda+A)^{-1}C \\ &\geq B(\lambda+A+C)^{-1}B + C(\lambda+A+B)^{-1}C\end{aligned}$$

One can see that

$$\begin{aligned}& (B+C)(\lambda+A)^{-1}(B+C) \geq B(\lambda+A+C)^{-1}B + C(\lambda+A+B)^{-1}C \quad (3)\\ \iff \quad & (\lambda+A+B+C)^{-1} - (\lambda+A+B)^{-1} \geq (\lambda+A+C)^{-1} - (\lambda+A)^{-1}. \quad (4)\end{aligned}$$

Indeed, multiply both sides of (4) by $\lambda+A+B+C$ to get (3). We show (4) yields (i).

Since f has an integral representation

$$f(t) = \alpha + \beta t + \int_0^\infty \left(\frac{\lambda}{\lambda^2+1} - \frac{1}{\lambda+t}\right) d\mu(\lambda),$$

where α is a real number, $\beta \geq 0$ and μ is a positive measure on $(0,\infty)$ such that $\displaystyle\int_0^\infty \frac{1}{\lambda^2+1} d\mu(\lambda) < \infty$,

$$\begin{aligned}& f(A+B+C) - f(A+B)\\ &= \beta C - \int_0^\infty \left\{(\lambda+A+B+C)^{-1} - (\lambda+A+B)^{-1}\right\} d\mu(\lambda)\\ &\leq \beta C - \int_0^\infty \left\{(\lambda+A+C)^{-1} - (\lambda+A)^{-1}\right\} d\mu(\lambda)\\ &= f(A+C) - f(A).\end{aligned}$$

We next show (ii).

First of all, assume $g(+0) < \infty$. Then by putting $g(0) = g(+0)$, $g(t)$ is operator convex on $[0, \infty)$. $f(t) := \frac{g(t)-g(0)}{t}$ is consequently operator monotone. By making use of the above representation of $f(t)$, we can obtain (ii). In this process we need $BC + CB \geq 0$. But we can derive it from (2).

We next consider the case where $g(+0)$ does not exist, i.e., $g(+0) = \infty$. Since $g(t+a)$ is operator convex on $[0, \infty)$ for every $a > 0$, by the above result

$$g(A+B+C+a) - g(A+B+a) \geq g(A+C+a) - g(A+a).$$

That $g(A)$ is bounded implies there is $m > 0$ so that $A \geq m$. Since $g(t+a)$ uniformly converges to $g(t)$ on $[m, \|A\|]$, $\|g(A+a) - g(A)\| \to 0$ as $a \to 0$. We therefore arrive at (ii). □

Remark 3.1. We note that each of the following is a sufficient condition for (2).

$$\begin{aligned} CB+BC &\geq 0, \quad AB = BA,\ AC = CA, && (5)\\ CB+BC &\geq 0, \quad AC^{-1}B + BC^{-1}A \geq 0, && (6)\\ CB+BC &\geq 0, \quad AB^{-1}C + CB^{-1}A \geq 0, && (7) \end{aligned}$$

where C and B are respectively assumed to be invertible in (6) and (7). Indeed, (5) is trivial; from (6) it follows that

$$(\lambda + A)C^{-1}B + BC^{-1}(\lambda + A) \geq 0$$

for every $\lambda > 0$, and hence

$$C^{-1}B(\lambda + A)^{-1} + (\lambda + A)^{-1}BC^{-1} \geq 0$$

for every $\lambda > 0$; this gives (2). The same argument yields from (7) to (2).

We now give the main theorem of this paper.

Theorem 3.3. *Let $A, B \geq 0$ and A be invertible. Then the following are equivalent to each other:*

(i) $AB + BA \geq 0$.

(ii) $Df(sA)(B)$ *is decreasing with respect to* $s > 0$ *for every operator monotone function* $f(t)$ *on* $(0, \infty)$.

(iii) $Dg(sA)(B)$ *is increasing with respect to* $s > 0$ *for every operator convex function* $g(t)$ *on* $(0, \infty)$.

Proof. We first show (i) implies that $Df(aA+bA)(B) \leq Df(aA)(B)$ for $a, b > 0$. Replace A, B, C in (7) with aA, bA, tB with $t > 0$, respectively. Then (7) is satisfied. By Lemma 3.2 we obtain

$$f(aA+bA+tB) - f(aA+bA) \leq f(aA+tB) - f(aA).$$

By dividing both side by t and taking the limits, we gain

$$Df(aA+bA)(B) \leq Df(aA)(B).$$

(iii) similarly follows (i). We show (ii) implies (i). From the assumption, for any $\lambda, s > 0$, we have

$$\begin{aligned}&(A+sA+\lambda)^{-1}B(A+sA+\lambda)^{-1}\\&\quad=\frac{1}{\lambda^2}Df_\lambda(A+sA)(B) \leq \frac{1}{\lambda^2}Df_\lambda(A)(B) = (A+\lambda)^{-1}B(A+\lambda)^{-1}\end{aligned}$$

since $f_\lambda(t) := \frac{\lambda t}{\lambda+t} = \lambda - \frac{\lambda^2}{\lambda+t}$ is operator monotone. From this it follows that for every $b > 0$

$$(A+sA+\lambda)^{-1}(B+b)(A+sA+\lambda)^{-1} \leq (A+\lambda)^{-1}(B+b)(A+\lambda)^{-1}.$$

By taking inverses of the both sides,we obtain

$$0 \leq sA(B+b)^{-1}(A+\lambda)+(A+\lambda)(B+b)^{-1}sA+sA(B+b)^{-1}sA,$$

and taking the limits $s \to 0$, we get

$$0 \leq A(B+b)^{-1}(A+\lambda)+(A+\lambda)(B+b)^{-1}A.$$

Divide the both sides by λ and take the limits to get $0 \leq A(B+b)^{-1}+(B+b)^{-1}A$. This entails $0 \leq AB+BA$. Assume (iii). Since $g(t) = t^2$ is operator convex and $Dg(A)(B) = AB+BA$, $s(AB+BA)$ is increasing on $0 < s < \infty$. This implies $AB+BA \geq 0$. □

We now combine the above theorem with Lemma 3.1.

Corollary 3.4. *Let $A, B \geq 0$ and A be invertible. Then the following are equivalent to each other:*

(i) $AB+BA \geq 0$.

(ii) *$Df(sA+s'B)(B)$ is decreasing with respect to both $s > 0$ and $s' \geq 0$ for every operator monotone function $f(t)$ on $(0,\infty)$.*

(iii) *$Dg(sA+s'B)(B)$ is increasing with respect to both $s > 0$ and $s' \geq 0$ for every operator convex function $g(t)$ on $(0,\infty)$.*

Proof. We show that (i) implies (ii). To do so, we first see $Df(sA+s'B)(B)$ is decreasing with respect to $s > 0$ for each fixed s'. For $a, b > 0$ replace A, B, C in (7) with $aA+s'B, bA, tB$ with $t > 0$, respectively. Then (7) is satisfied. Because of Lemma 3.2 we can get

$$Df\left(aA+bA+s'B\right)(B) \leq Df\left(aA+s'B\right)(B).$$

This is the desired result. As we mentioned in Introduction $f(t)$ is operator concave. $-f(t)$ is therefore operator convex. By Lemma 3.1, for each s, $Df(sA+s'B)(B)$ is decreasing with respect to $s' \geq 0$. We consequently get (ii). The same argument leads from (i) to (iii). The rest has been already shown in Theorem 3.3. □

Corollary 3.5. *Let $A \geq 0$ be invertible. Then we have*

(i) *A commutes to $B \geq 0$ if and only if $Dg(sA)(B^n)$ is increasing on $0 < s < \infty$ for every natural number n and for every operator convex function $g(t)$ on $(0, \infty)$.*

(ii) *A is constant if and only if $Dg(sA)(B)$ is increasing on $0 < s < \infty$ for every $B \geq 0$ and for every operator convex function $g(t)$ on $(0, \infty)$.*

Proof. By Theorem 3.3, $Dg(sA)(B^n)$ is increasing with respect to $s > 0$ if and only if $B^n A + AB^n \geq 0$ for every n. That is equivalent to A commutes with B (see [10]).

(ii) is obvious by using (i). □

On Corollary 3.5, if we replace operator convex function with operator monotone function, we get similar result.

We finally extend Theorem 3.3.

Theorem 3.6. *Let $A, B, C \geq 0$ and A be invertible. Then the following are equivalent:*

(i) $AC + CA \geq 0$, $BA^{-1}C + CA^{-1}B \geq 0$.

(ii) *$Df(sA+B)(C)$ is decreasing on $0 < s < \infty$ for every operator monotone function $f(t)$ on $(0, \infty)$.*

(iii) *$Dg(sA + B)(C)$ is increasing on $0 < s < \infty$ for every operator convex function $g(t)$ on $(0, \infty)$.*

Proof. Assume (i). Then for $a, b > 0$ replace A, B, C in (7) with $aA+B, bA, tC$ with $t > 0$, respectively. Then (7) is satisfied. By (i) of Lemma 3.2, we have

$$f(aA + B + bA + tC) - f(aA + B + bA) \leq f(aA + B + tC) - f(aA + B)$$

for every operator monotone function $f(t)$ on $(0, \infty)$. We therefore obtain

$$Df(aA + bA + B)(C) \leq Df(aA + B)(C).$$

This means (ii). (iii) can be similarly derived from (i). We next show (ii) implies (i). From the assumption, for any $0 < s' < s$ and $\lambda > 0$, we have

$$\begin{aligned}
&(sA + B + \lambda)^{-1}C(sA + B + \lambda)^{-1} \\
&= \frac{1}{\lambda^2}Df_\lambda(sA + B)(C) \leq \frac{1}{\lambda^2}Df_\lambda (s'A + B) (C) \\
&= (s'A + B + \lambda)^{-1} C (s'A + B + \lambda)^{-1},
\end{aligned}$$

where $f_\lambda(t) = \frac{\lambda t}{\lambda+t} = \lambda - \frac{\lambda^2}{\lambda+t}$. By letting $s' \to 0$, we have

$$(sA + B + \lambda)^{-1}C(sA + B + \lambda)^{-1} \leq (B + \lambda)^{-1}C(B + \lambda)^{-1},$$

Multiply the both sides by $sA+B+\lambda$,

$$0 \le sA(B+\lambda)^{-1}C + sC(B+\lambda)^{-1}A + s^2A(B+\lambda)^{-1}C(B+\lambda)^{-1}A.$$

This gives

$$0 \le A(B+\lambda)^{-1}C + C(B+\lambda)^{-1}A.$$

Since $\lambda > 0$ is arbitrary, one can easily derive (i) from this. To show that (iii) implies (i), consider the operator convex functions

$$g_\lambda(t) = \frac{\lambda t^2}{\lambda + t} = \lambda t - \lambda f_\lambda(t)$$

for $\lambda > 0$. Since

$$Dg_\lambda(sA+B)(C) = \lambda C - \lambda Df_\lambda(sA+B)(C),$$

$Df_\lambda(sA+B)(C)$ is decreasing on $0 < s < \infty$. The above argument leads us to (i). □

We note that the case of $B = 0$ in this theorem is the same as Theorem 3.3.

Corollary 3.7. *Let $A, B, C \ge 0$ and A, B be invertible. Then the following are equivalent:*

(i) *$AC + CA \ge 0$, $BC + CB \ge 0$ and $BA^{-1}C + CA^{-1}B \ge 0$.*

(ii) *$Df(sA + s'B)(C)$ is decreasing on $0 < s < \infty$ and $0 < s' < \infty$ for every operator monotone function $f(t)$ on $(0, \infty)$.*

(iii) *$Dg(sA + s'B)(C)$ is increasing on $0 < s < \infty$ and $0 < s' < \infty$ for every operator convex function $g(t)$ on $(0, \infty)$.*

Proof. Assume (i). Since $BA^{-1}C + CA^{-1}B \ge 0$ is equivalent to $AB^{-1}C + CB^{-1}A \ge 0$, by Theorem 3.6, we get (ii) and (iii). The converse statements have been also shown there. □

As $C = B$ in Corollary 3.7 , the condition (i) reduces to $AB + BA \ge 0$. This says that Corollary 3.7 is an extension of Corollary3.4.

References

[1] T. Ando, X.Zhan, Norm inequalities related to operator monotone functions, Math. Ann. 315 (1999) 771-780.

[2] R. Bhatia, Matrix Analysis, 1996.

[3] J. C. Bourin and M. Uchiyama, A matrix subadditivity inequality for f(A+B) and f(A)+f(B), Linear Algebra Appl. 423 (2007), 512–518.

[4] M. D. Choi, A Schwarz inequality for positive linear maps on C^*-algebras, Ill. J. Math. 18. (1974)565–574.

[5] C. Davis, A Schwarz inequality for convex operator functions, Proc. Amer. Math. Soc. 8(1957) 42–44.

[6] F. Hansen, G. K. Pedersen, Jensen's inequality for operators and Löwner's theorem, Math. Ann., 258 (1982), 229–241.

[7] F. Hiai, T. Sano, Loewner matrices of matrix convex and monotone functions. J. Math. Soc. Japan 64 (2012), no. 2, 343–364.

[8] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge Universtiy Press, 1991.

[9] M. S. Moslehian and H. Najafi, Around operator monotone functions. Integral Equations Operator Theory 71 (2011) 575–582.

[10] M. Uchiyama, Commutativity of selfadjoint operators. Pacific J. Math. 161 (1993) 385–392.

[11] M. Uchiyama, Powers and commutativity of selfadjoint operators, Math. Ann. 300 (1994) 643–647.

[12] M. Uchiyama, Inverse functions of polynomials and orthogonal polynomials as operator monotone functions, Trans. Amer. Math. Soc. 355(2003) 4111–4123.

[13] M. Uchiyama, Operator monotone functions and operator inequalities [translation of Sugaku 54 (2002), no. 3, 265–279; MR1929896]. Sugaku Expositions. 18 (2005), no. 1, 39–52.

[14] M. Uchiyama, Subadditivity of eigenvalue sums, Proc. Amer. Math. Soc. 134 (2006)1405–1412.

[15] M. Uchiyama, Operator monotone functions, positive definite kernels and majorization, Proc. Amer. Math. Soc. 138 (2010)3985–3996.

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